Error Analysis of Two New Gradiometer-Aided Inertial Navigation Systems

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Two Kalman filtering integration techniques are presented for using gradiometer measurements to compensate an inertial navigation system for gravity disturbances. Both approaches use knowledge of the reference ellipsoid gravity field in addition to gradiometer data and lead to bounded residual gravity-induced navigation errors. Although the two compensation techniques are similar, one appears to offer significant theoretical and practical advantages. The mechanization and error equations for each technique are outlined and the significance of various terms which appear in these equations is discussed in the context of gradiometer-aided inertial navigation.

Nomenclature

$d/dt(\ )_S$	= time derivative taken in the S reference frame (error equation solution frame)
δR	= inertial navigation system position error
δV	= inertial navigation system velocity error
δV_r	= velocity reference error
V	= vehicle velocity vector with respect to the Earth (groundspeed)
A	= specific force vector, i.e., vehicle acceleration due to all forces acting except gravity (ideal accelerometer output)
ψ	= small angle misalignment between platform and computer reference frames
ϵ	= total gyro drift rate due to all gyro error sources
ω_{IS}	= angular rate of error equation solution frame with respect to inertial space
ω_{ES}	= angular rate of error equation solution frame with respect to the Earth
Ω	= Earth's angular rate with respect to inertial space
δA	= total accelerometer error vector (difference between ideal and actual accelerometer output)
K	= velocity damping gain(matrix)
δg	= total error in INS gravity calculation

I. Introduction

GRAVITY uncertainties are an inexorable source of error in all inertial navigation systems and are particularly important in modern, high quality equipment. One promising approach for mitigating gravity-induced navigation errors is to measure the Earth's gravity gradient field with a gravity gradiometer and to use these measurements to reduce the gravity uncertainty and its corresponding effect on navigation system errors. Recent advances in gradiometer technology indicate that real-time gradiometric compensation of inertial systems for deflections of the vertical and gravity anomalies may be possible soon.

In principle, real-time deflections and anomalies can be found by continuous integration of measured gravity gradients. However, because straightforward integration of the gradients transforms gradiometer errors into growing errors in the recovered gravity vector, the direct or "open-loop" integration approach is unattractive. Although some modified integration schemes for processing gradiometric data have been proposed, which result in stable gravity errors, difficulties remain with unbounded growth in one or more of the navigation variables (e.g., Ref. 1).

However, gravity errors need not lead to growing navigation errors. In a conventional (no gradiometer), velocity-aided inertial system, the gravity-induced errors are stable. Hence it is appropriate to seek gradiometer/inertial configurations which embody available knowledge of the global (long-wavelength) characteristics of the Earth's gravity field in a manner similar to the conventional Inertial Navigation System (INS) gravity mechanization shown in Fig. 1. The approach taken in this paper is to augment the gradiometer measurements of the local (short wavelength) gravity field with the long-wavelength information provided by the reference ellipsoid gravity formula. Two new techniques are presented for integrating gravity gradients, which avoid instability of the residual navigation and gravity estimation errors. Error propagation equations for each technique are derived and discussed. Both algorithms optimally process gradiometric data with a Kalman filter (or smoother) as in Ref. 2.

Two limitations of the present paper should be mentioned. First, it is not necessary to use a Kalman filter, as proposed herein, to achieve bounded errors in a gradiometer-aided INS. Continuous feedback INS mechanizations normally are used to incorporate velocity and altitude aids, and similar continuous feedback loops can be used for gradiometric damping. The design of such a "conventionally-damped" (as opposed to "Kalman damped") gradiometer-aided INS is not considered herein. Second, the performance of a gradiometer-aided INS can be improved by simultaneously optimizing gradiometer and velocity damping. For example, if both gradiometric and velocity reference data are fed into the Kalman filter, the filter will (theoretically) provide optimal damping for both aids. In the present paper, the Kalman filter performs only gradiometric damping; velocity damping is achieved using conventional feedback loops, and the velocity feedback gain is fixed at a constant value.

II. General INS Vector Error Equations

Navigation errors typically are divided into two categories: long-term "Earth loop" errors arising mainly from gyro errors and short-term "Schuler loop" errors resulting from accelerometer and gravity compensation errors. This simplification is possible because of the well-known decoupling property of the INS error equations,[3,4] namely that the Earth

Presented as Paper 75-1091 at the AIAA Guidance and Control Conference, Boston, Mass., Aug. 20-22, 1975; submitted Sept. 4, 1975; revision received Feb. 2, 1976. This paper is based on work done at The Analytic Sciences Corporation, under Contract No. DMA700-74-C-0075 for the Defense Mapping Agency Aerospace Center, St. Louis, Mo.

Index categories: Navigation, Control, and Guidance Theory; Aircraft Navigation, Communication, and Traffic Control.

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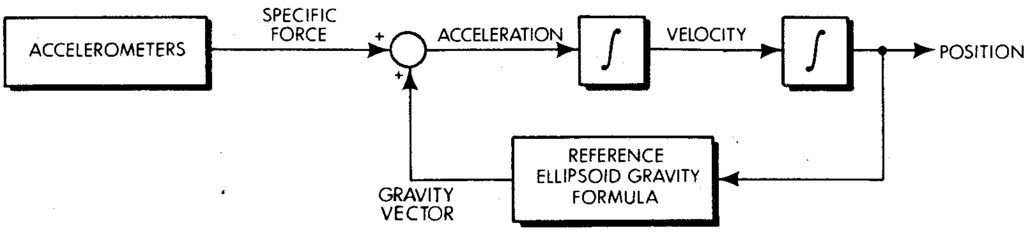

Fig.1 Gravity computation in a conventional inertial navigation system.

loop errors "drive" the Schuler loop errors. Because, as will be demonstrated subsequently, gravity and gradiometer errors affect only the Schuler loop errors, it is convenient and appropriate to ignore the Earth loop errors.‡

The error equations for a conventional velocity- and altitude-damped INS are introduced. These equations are well known[3-5] and, except to define notation, are not discussed in detail here. Velocity and altitude damping is assumed so that the INS Schuler loop errors are stable.[6] Although more complex damping configurations could be assumed, no advantage would be gained for the purpose of this discussion.

The INS error equations are

$$\mathrm{d}/\mathrm{d}t(\delta \boldsymbol{R})_S = \delta \boldsymbol{V} - \omega_{ES} \times \delta \boldsymbol{R} \tag{1}$$

$$\mathrm{d}/\mathrm{d}t\ (\delta \boldsymbol{V})_S = \delta \boldsymbol{A} - \psi \times \boldsymbol{A} + \delta \boldsymbol{g} - (\omega_{IS} + \Omega) \times \delta \boldsymbol{V} - K(\delta \boldsymbol{V} + \psi \times \boldsymbol{V} - \delta \boldsymbol{V}_r) \tag{2}$$

$$\mathrm{d}/\mathrm{d}t(\psi)_S = -\omega_{IS} \times \psi + \epsilon \tag{3}$$

Note that the assumption of altitude damping affects the behavior of the computed gravity error δg and does not appear explicitly in Eqs. (1-3). Details may be found in Ref. 6. Two comments are appropriate in regard to Eqs. (1-3) before proceeding with the gradiometer development:

1) The formulation is completely general and involves no assumptions about mechanization (local level, space-stable, strapdown, etc.).

2) No navigation error terms (δV, δR) gravity errors δg, or acceleration errors δA appear in Eq. (3). Hence the dynamics of Eq. (3) (Earth loop errors) operate independently of gravity errors.

III. Gradiometer as an External Navigation Aid Mechanization

One method for introducing gravity gradiometer measurements into a multisensor inertial navigation system is to configure the gradiometer as an external navigation aid (GAEA) to the system as depicted in Fig. 2. Other external sensors are: an external velocity reference to provide Schuler loop damping, an altimeter (or depth gauge) for vertical channel stabilization, and any additional navigation aids which are desirable or appropriate. For clarity, external navigation aids other than the gradiometer have been omitted from Fig. 2.

Note that the GAEA mechanization is implemented without modifying the inertial configuration of Fig. 1; the gradiometer and Kalman filter are simply "add-ons." It is particularly convenient to express the filter input as the product of the gravity gradient tensor and the velocity vector. This obviates the need for tensor models in the Kalman filter, which would be required if the difference gravity gradients were input directly into the filter. The filter provides estimates of the gravity disturbance vector, and the gravity-induced velocity and position errors. These estimates are used to compensate the INS outputs.§ Because the gradiometer signal never enters the acceleration and velocity integrators of the inertial system, the position and velocity error equations [Eqs. (1) and (2)] are unaffected by the gradiometer. Hence, gradiometer errors propagate directly through the Kalman filter into the corrected navigation variables. This signal flow avoids instability in the navigation errors, which may be experienced when gradiometer signals are introduced directly into the inertial system. The Kalman filter provides a particularly effective means for processing differences between the gravity gradient signals from the gradiometer and reference gradients generated from the reference ellipsoid formula.

There is one possible drawback of the GAEA mechanization (Fig. 2) that should be mentioned. If the vehicle is moving slowly (say, less than 1 knot), then velocity error is comparable in magnitude to total velocity. As a consequence, one is faced with a nonlinear filtering problem, and the usual Kalman filtering equations do not apply. Note, too, that, in the case of zero vehicle velocity, the gravity gradients are indistinguishable from an uncalibrated gradiometer bias. Such difficulties are not considered further.

Error Equations

The measurement error equation for the GAEA mechanization now is derived. The gravity vector $\boldsymbol{g}$ is a function of the vehicle position vector $\boldsymbol{R}$

$$\boldsymbol{g} = \boldsymbol{g}(\boldsymbol{R}) \tag{4}$$

Differentiation of this equation yields

$$(\mathrm{d}/\mathrm{d}t)\ (\boldsymbol{g})_E = (\nabla \boldsymbol{g})(\mathrm{d}/\mathrm{d}t)\ (\boldsymbol{R})_E \tag{5}$$

where ∇ is the gradient operator and the subscript E indicates that the differentiation is being carried out in an Earth-fixed reference frame. By definition, the gradient of gravity is the gravity gradient tensor Γ

$$\Gamma = \nabla \boldsymbol{g} \tag{6}$$

Also, by definition, the time-rate-of-change of vehicle position, as seen in an Earth-fixed frame, is equal to "groundspeed," V

$$\boldsymbol{V} = (\mathrm{d}/\mathrm{d}t)\ (\boldsymbol{R})_E \tag{7}$$

Equations (4-7) yield the result

$$(\mathrm{d}/\mathrm{d}t)\ (\boldsymbol{g})_E = \Gamma \boldsymbol{V} \tag{8}$$

The navigation, gravity, and gravity gradient errors are defined by the following equations

$$\boldsymbol{R}_C = \boldsymbol{R} + \delta \boldsymbol{R} \tag{9}$$

‡When a Kalman filter is used to effect INS position updates, Earth loop and Schuler loop errors become coupled through the filter dynamics. This more complicated situation is not analyzed herein.

§Of course, in contrast to the feedforward scheme discussed here, the Kalman filter could be mechanized in a feedback fashion. Although the two mechanizations provide theoretically equivalent results, the feedforward mechanization is being considered for the purpose of conceptual clarity.

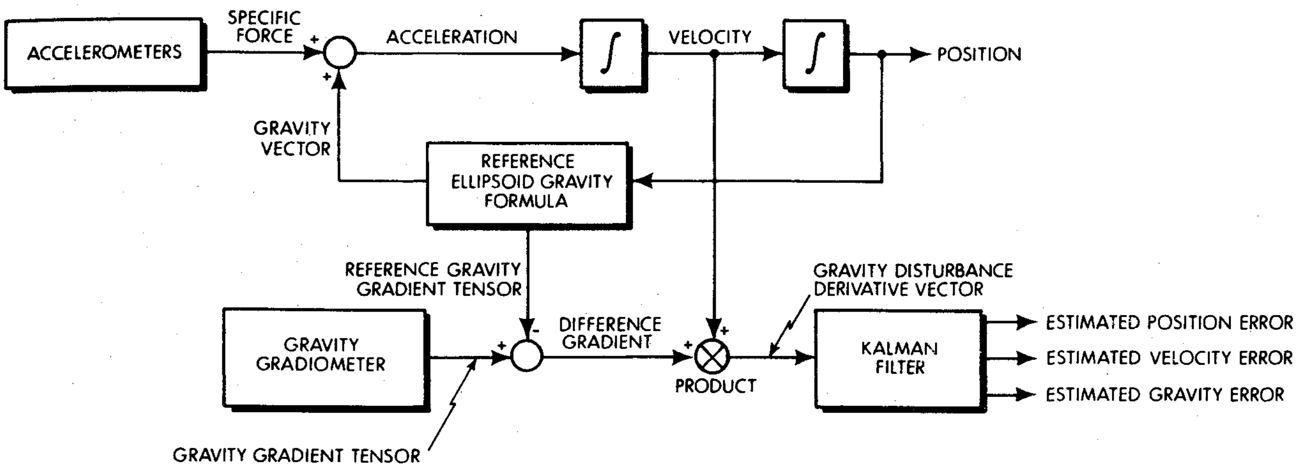

Fig.2 Gradiometer as an external navigation aid (GAEA).

$$V_C = V + \delta V \tag{10}$$

$$\Gamma_C = \Gamma + \delta\Gamma \tag{11}$$

where R_C and V_C are the position and velocity computed by the INS, and Γ_C is the gravity gradient furnished by the gradiometer. It is convenient to distinguish two sources of error which affect the reference gravity vector g_{RE} and the reference gradient Γ_{RE}, namely, errors in gravity because the reference field is only an approximation of the true field and errors caused by an incorrect position being used in the computation:

$$g_{RE} = g + \Delta g + \epsilon g_{RE}(\delta R) \tag{12}$$

$$\Gamma_{RE} = \Gamma + \Delta\Gamma + \epsilon\Gamma_{RE}(\delta R) \tag{13}$$

The terms Δg and $\Delta\Gamma$ represent the errors inherent in the reference gravity (e.g., ellipsoid) approximation; the terms ϵg_{RE} and $\epsilon\Gamma_{RE}(\delta R)$ correspond to errors induced by position error.

In the GAEA mechanization (Fig. 2), the difference between the gradiometer measurement and the reference gradient is multiplied by velocity and processed by the Kalman filter algorithm. Hence, the state space measurement y on which the filter operates is

$$y = (\Gamma_C - \Gamma_{RE}) V_C \tag{14}$$

Substituting Eqs. (10-11 and 13) into Eq. (14) provides the relation

$$y = [\delta\Gamma - \Delta\Gamma - \epsilon\Gamma_{RE}(\delta R)]V + [\delta\Gamma - \Delta\Gamma - \epsilon\Gamma_{RE}(\delta R)]\delta V \tag{15}$$

It is convenient to expand the reference ellipsoid errors (position induced) in a generalized Taylor's series

$$\epsilon g_{RE}(\delta R) = \nabla g_{RE}\delta R + (1/2!)\nabla^2 g_{RE}(\delta R)^2 + \ldots\text{H.O.T.} \tag{16}$$

$$\epsilon\Gamma_{RE}(\delta R) = \nabla\Gamma_{RE}\delta R + (1/2!)\nabla^2\Gamma_{RE}(\delta R)^2 + \ldots\text{H.O.T.} \tag{17}$$

Under the usual perturbation assumptions,

$$\delta R << R \tag{18}$$

$$\delta V << V \tag{19}$$

terms higher than the first order may be neglected in Eqs. (15-17). Hence Eq. (15) becomes

$$y = \delta\Gamma V - \Delta\Gamma V - \nabla\Gamma_{RE}\delta R V \tag{20}$$

where Eq. (17) has been truncated to a single term. Equations (1-3),¶ (20), and the Kalman filter equations provide the complete set of error propagation relations for describing a GAEA-configured INS. The significance of the individual terms of Eq. (20) now is reviewed. The second quantity to the right of the equal sign ($\Delta\Gamma V$) corresponds to the gravity disturbance vector: the error made in approximating the Earth's gravity field by the reference ellipsoid gravity field. As such, it is the desired portion of the measurement (signal). The $\delta\Gamma V$ term, arising from errors in the measured gradient, is a degrading term (gradiometer noise). A second measurement noise term appears ($\nabla\Gamma_{RE}\delta R V$) because incorrect values of position are used in the reference ellipsoid formula. Note that $\nabla\Gamma_{RE}$ is a tensor quantity of order three. This "double gradient" error term is usually small in high-quality inertial systems. Its value corresponds to gradiometer error on the order of 1.3 EU** per nautical mile of position error.

A form of Eq. (20) convenient for computer simulation may be obtained by repeating the derivation of Eq. (8) for the gravity disturbance. As with Eq. (4),

$$\Delta g = \Delta g(R) \tag{21}$$

Hence,

$$(\mathrm{d}/\mathrm{d}t)(\Delta g)_E = (\nabla g)(\mathrm{d}/\mathrm{d}t)(R)_E \tag{22}$$

Analogous to the definition of Γ in Eq. (6),

$$\Delta\Gamma = \nabla(\Delta g) \tag{23}$$

Substituting Eqs. (7) and (23) into Eq. (22) provides the relation

$$(\mathrm{d}/\mathrm{d}t)(\Delta g)_E = \Delta\Gamma V \tag{24}$$

Inserting Eq. (24) into the Kalman filter measurement Eq. (20) yields

$$y = \delta\Gamma V - (\mathrm{d}/\mathrm{d}t)(\Delta g)_E - \nabla\Gamma_{RE}\delta R V \tag{25}$$

The advantage offered by the form of Eq. (25) is that simple gravity disturbance vector models may be used with Eq. (25), whereas in Eq. (20), modeling of the gravity disturbance gradient tensor is required.

¶The equations associated with the Kalman filter itself are well known and will not be repeated here. A complete treatment is given in Ref. 7.

**One Eotvos Unit (EU) = 10^{-9} sec^{-2}.

Local-Level Form of the Measurement Equation

In arbitrary-azimuth, local-level coordinates (X, Y, Z down), Eq. (25), written out in component form, is

$$y_X = \delta\gamma_{XX} v_X + \delta\gamma_{XY} V_Y + \delta\gamma_{XZ} V_Z - \Delta\dot{g}_X - (3g/R^2)(V_Z \delta R_X + V_X \delta R_Z) \tag{26}$$

$$y_Y = \delta\gamma_{XY} V_X + \delta\gamma_{YY} V_Y + \delta\gamma_{YZ} V_Z - \Delta\dot{g}_Y - (3g/R^2)(V_Y \delta R_Z + V_Z \delta R_Y) \tag{27}$$

$$y_Z = \delta\gamma_{XZ} V_X + \delta\gamma_{YZ} V_Y + \delta\gamma_{ZZ} V_Z - \Delta\dot{g}_Z - (3g/R^2)(V_X \delta R_X + V_Y \delta R_Y - 2V_Z \delta R_Z) \tag{28}$$

where the subscripts identify components in the subscripted direction, and the symmetry of the gravity gradient tensor has been invoked. The lower case $\delta\gamma_{ij}$ indicates the ij element of the tensor $\delta\Gamma$, and the overhead dot signifies a time derivative. The quantity R indicates the Earth's radius. For X, Y taken as north and east, consideration of the north channel only ($V_E = V_Z = \delta R_Z = 0$) reduces Eq. (26) to

$$y_N = \delta\gamma_{NN} V_N - \Delta\dot{g}_N \tag{29}$$

IV. Reference Ellipsoid as an External Aid Mechanization

An obvious permutation of the GAEA mechanization is the use of the gradiometer as the primary gravity reference for the inertial system. The gravity information provided by the reference ellipsoid gravity formula then is used as an external navigation aid (REFAEA). A functional diagram is presented in Fig. 3; again, for clarity, the only external sensor shown is the gradiometer.

Because the GAEA and REFAEA mechanizations employ the same essential elements (INS, gradiometer, Kalman filter), one might think that the residual gravity and navigation estimation errors in these two configurations are identical. It turns out that, in general, this is not the case. For this reason, both mechanizations have been investigated in detail to determine if either enjoys an advantage. A comparison is given in Sec. V.

Error Equations

Note that the Kalman filter measurement is the same as for the GAEA mechanization and is given by Eq. (14). Hence, the measurement error equation also is the same as that for the GAEA mechanization, Eq. (20). However, because the reference value of gravity supplied to the inertial system is derived from the gradiometer instead of from the reference ellipsoid, the error propagation of the INS Schuler loop variables is no longer given solely by Eqs. (1) and (2). More insight into the form of the modified INS error equations is provided by disregarding those signal paths in Fig. 3 which do not return information to the acceleration, velocity or position variables, i.e., lead only to the filter. When this is done, the "gradiometer-alone" INS configuration of Fig. 4 is obtained. Note that in the gradiometer-alone mechanization, the gravity error δg is no longer a function of position. Instead, it propagates according to the relation derived in Ref. 8

$$(\mathrm{d}/\mathrm{d}t)(\delta \boldsymbol{g})_C = \Gamma \delta \boldsymbol{V} + \delta\Gamma \boldsymbol{V} - \boldsymbol{\omega}_{EC} \times \delta \boldsymbol{g} - \delta\boldsymbol{\omega}_{EC} \times \boldsymbol{g} \tag{30}$$

where C denotes the computer frame. In the present analysis, the position and velocity outputs of the inertial system define the computer frame relative to the Earth-fixed frame; hence, the computer has perfect knowledge of ω_{EC} and $\delta\omega_{EC}$ is identically zero. Equation (30) is transformed into the error equation solution frame S by the Coriolis conversion

$$(\mathrm{d}/\mathrm{d}t)(\delta \boldsymbol{g})_S = (\mathrm{d}/\mathrm{d}t)(\delta \boldsymbol{g})_C + \boldsymbol{\omega}_{SC} \times \delta \boldsymbol{g} \tag{31}$$

In the S frame, Eq. (30) becomes

$$(\mathrm{d}/\mathrm{d}t)(\delta \boldsymbol{g})_s = \Gamma \delta \boldsymbol{V} + \delta\Gamma \boldsymbol{V} - \boldsymbol{\omega}_{ES} \times \delta \boldsymbol{g} \tag{32}$$

Equations (1-3) and Eq. (32) describe error propagation in the gradiometer-alone mechanization. When augmented with Eq. (20) and the Kalman filter equations, the error propagation relations for the REFAEA mechanization are obtained.

Note that no position error term appears in Eq. (32); hence, there is no coupling from the position error $\delta\boldsymbol{R}$ of Eq. (1) to Eq. (32). As a result, in the gradiometer-alone and REFAEA configurations, the integration of Eq. (1) is "open-loop" and, for bias or broadband gradiometer errors, results in unbounded position errors. This behavior is in contrast to the error propagation in a conventional inertial system or in the GAEA configuration where the integration of all navigation errors is "closed-loop."

Local-Level Form of the Gravity Error Equation

For a spherical Earth, the gravity gradient matrix in local-level (north, east, down) coordinates is given by

$$\Gamma = \begin{bmatrix} -\omega_s^2 & 0 & 0 \\ 0 & -\omega_s^2 & 0 \\ 0 & 0 & 2\omega_s^2 \end{bmatrix} \tag{33}$$

where

$$\omega_s^2 = g/R \tag{34}$$

The vector $\boldsymbol{\omega}_{ES}$ is given by

$$\omega_{ES} = \begin{bmatrix} V_E/R \\ -V_N/R \\ -(V_E \tan L)/R \end{bmatrix} \tag{35}$$

where L is vehicle latitude. Equation (32) becomes

$$\delta\dot{g}_N = (V_N/R)\delta g_Z - (V_E \tan L/R)\delta g_E - \omega_s^2 \delta V_N + e_N \tag{36}$$

$$\delta\dot{g}_E = (V_E \tan L/R)\delta g_N + (V_E/R)\delta g_Z - \omega_s^2 \delta V_E + e_E \tag{37}$$

$$\delta\dot{g}_Z = -(V_E/R)\delta g_E - (V_N/R) - (V_N/R)\delta g_N + 2\omega_s^2 \delta V_Z + e_Z \tag{38}$$

where $\boldsymbol{e}_N$, $\boldsymbol{e}_E$, and $\boldsymbol{e}_Z$ are components of the vector $\boldsymbol{e}$, given by

$$\boldsymbol{e} = \delta\Gamma \boldsymbol{V} \tag{39}$$

Local-level formulations of the remaining applicable navigation error equations, Eqs. (1-3), have been treated extensively in the literature (Refs. 3, 5, and 9) and will not be repeated here. It is interesting however to observe the form taken by Eqs. (1, 2, and 36), if only the north Schuler loop channel of the INS is considered. The appropriate equations are

$$\delta\dot{g}_N = -\omega_s^2 \delta V_N + e_N \tag{40}$$

$$\delta\dot{V}_N = \delta g_N + \delta A_N + A_E \psi_Z \tag{41}$$

$$\delta\dot{R}_N = \delta V_N \tag{42}$$

where the cross-coupled Earth Loop (ψ) driving term is indicated in Eq. (41). A block diagram of these equations is

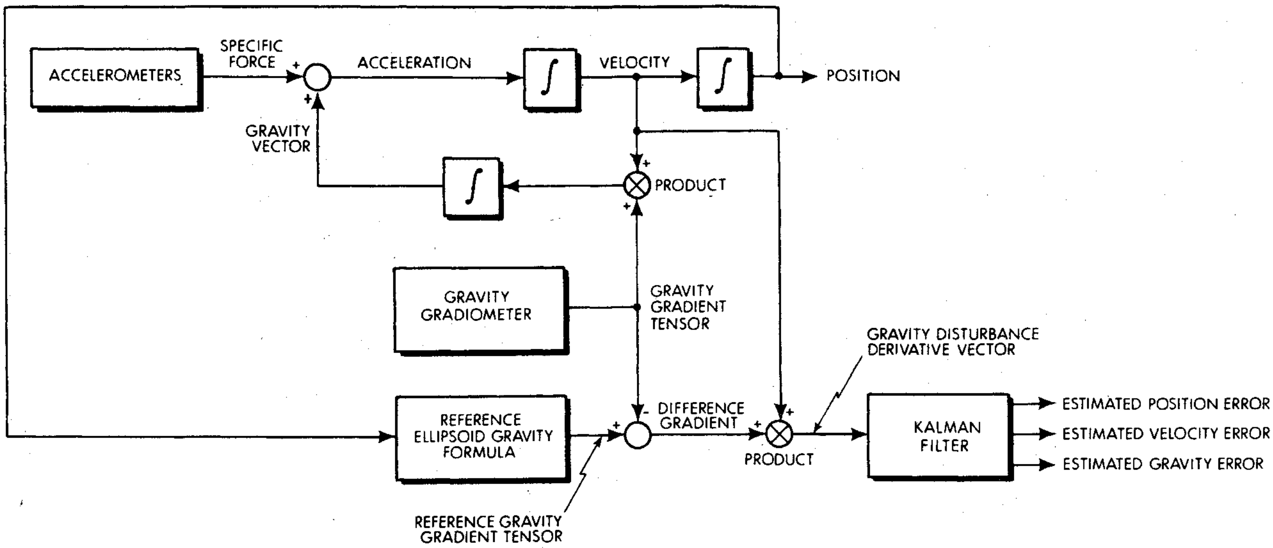

Fig.3 Reference ellipsoid formula as an external navigation aid (REFAEA).

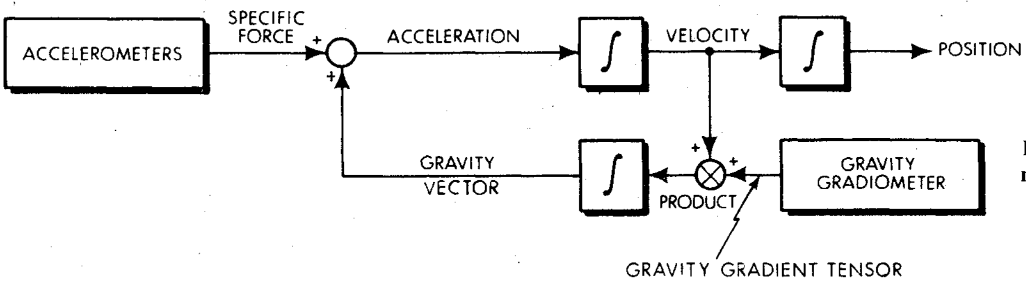

Fig.4 Gradiometer-alone gravity mechanization.

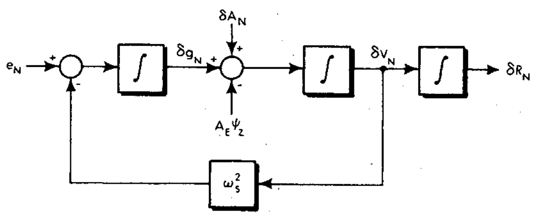

Fig.5 Error diagram for the north channel of a gradiometer-aided inertial navigation system (gradiometer-alone configuration).

presented in Fig. 5. Note the open-loop integration of the velocity error inherent in the gradiometer-alone configuration.

V. Comparison of the GAEA and REFAEA Mechanizations

Residual Errors

At first examination, it might be expected that, although the INS errors are different in the GAEA and REFAEA configurations, the residual errors (remaining after the filter corrections have been applied) are the same. Such an intiutive guess based on the "same information available to the filter" in each mechanization turns out to be almost correct.

Comparison of these errors is made by writing matrix error covariance equations for the residual errors of the "feedforward" Kalman filter GAEA mechanization of Fig. 2, i.e., for the autocovariances and cross-covariances of the difference between the navigation system error states and the filter estimates of those error states. Another set of covariance equations is written for the REFAEA mechanization of Fig. 3. Both sets of equations assume the use of an optimum Kalman filter and correct statistical modeling of the gravity process. As the manipulations are quite lengthy, it is not appropriate to detail them here. Instead the highlights of the procedure are given, along with conclusions. For the GAEA mechanization the state Eqs. (1) and (2) are used. This same set of state equations, as well as Eq. (32), are used for the REFAEA mechanization. The measurement equation, Eq. (25) is the same for both mechanizations. Residual estimation error covariance expressions for the REFAEA mechanization take on an increased dimensionality (vis-a-vis the GAEA error covariance expressions) because of the additional state equation [Eq. (32)]. In addition, the Kalman filter formulation for correlated process and measurement noise is required to express the error covariances of the REFAEA mechanization. Error covariance equations of the usual state-space form (Ref. 7)

$$\dot{P}=FP + PF^T + Q-PH^TR^{-1}HP \tag{43}$$

are derived for each mechanization, where P denotes the error covariance matrix for all system states; F is the dynamics matrix of the state equation, and H is the measurement matrix. The matrices Q and R are appropriately-defined "process noise" and "measurement noise" spectral densities. The superscript T denotes matrix transpose. The equations resulting from the expansion of Eq. (43) for each mechanization then are manipulated (i.e., transformed) so that they "match," term-for-term, as closely as possible. From the final matched form of the equations that is obtained the following inferences are drawn.

1) Residual errors in estimated gravity are identical for both mechanizations, i.e., the error covariance equations match, term-for-term.

2) If terms of the form $\omega_{ES}\times(\ldots)$ in Eqs. (1) and (2) are neglected, (e.g., in single axis INS representations, residual navigation errors in position and velocity are identical for both mechanizations.

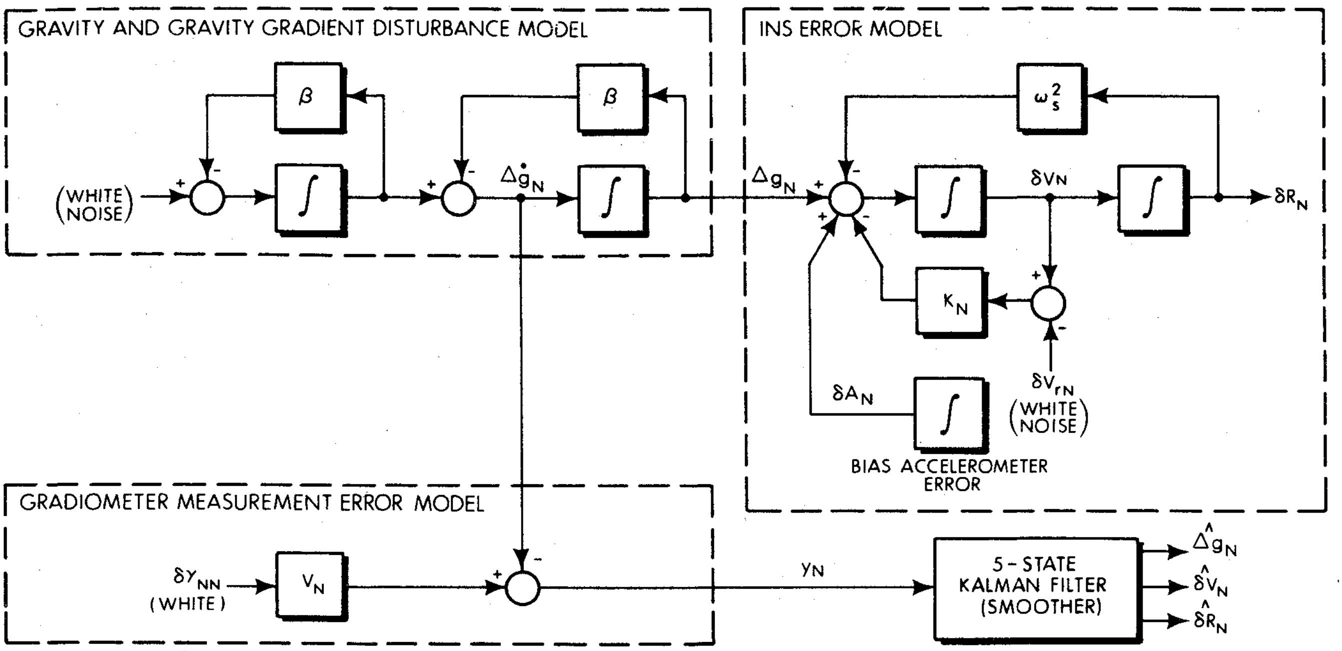

Fig.6 Optimal filter processing of gradiometer data.

3) In general, residual position and velocity errors are different for the two mechanizations. However, this difference is a function only of cross coupled INS error terms in ω_{ES}, terms which typically are quite small. Hence, the residual error magnitudes of the GAEA and REFAEA mechanizations are "essentially the same."

Stability

As has already been noted, the Schuler loop navigation errors of a conventional velocity- and altitude-damped INS are stable. Also, the state equations modeled in the Kalman filter are observable and controllable; hence (Ref. 7) the GAEA mechanization leads to stable residual errors for all navigation variables (in the absence of gyro drift). Since the residual errors of an optimal Kalman filter are identical for both the feedforward filter mechanization discussed earlier and for a "feedback" filter mechanization (Ref. 7), the option for either the feedforward or feedback model of filter implementation can be exercised on the basis of hardware considerations.

However, because of the open-loop integration associated with Eq. (1) in the REFAEA mechanization (also see Fig. 4), INS position errors because of broadband noise or bias gradiometer errors are unbounded. Yet, because differences between the GAEA and REFAEA errors are small, the residual position errors will be stable at approximately the same values as would be provided by the GAEA mechanization. Hence, the REFAEA mechanized Kalman filter in the feedforward configuration tracks the "growing INS position errors' by producing growing estimates of the position error. Since such behavior is unattractive in practical systems, any realization of the REFAEA gradiometer mechanization would require the Kalman filter to be mechanized in the feedback mode to prevent growing position errors. Nonetheless, there still could be serious problems of convergence in the filter algorithm because of modeling errors or a necessary reduction in the number of state variables carried in a vehicle's onboard computer. Successful implementation of the REFAEA mechanization most likely would require a carefully selected, suboptimal, direct feedback stabilization scheme.

The GAEA gradiometer mechanization thus is preferred over the REFAEA mechanization. Its advantages are summarized below.

1) No intermediate variables in the GAEA mechanization are unbounded, but growing position errors and position error estimates in the REFAEA mechanization require the additional complexity of feedback stabilization.

2) The Kalman filter may be implemented in either a feedback or a feedforward configuration in the GAEA mechanization.

3) The GAEA mechanization requires minimal change in the programming of a conventional speed- and altitude-damped INS. Should the gradiometer signal become unavailable, the INS reverts to the conventional gravity configuration depicted in Fig. 1.

VI. Computer Simulation Results

Covariance simulations were conducted for a single INS axis aided by a gradiometer in the GAEA or REFAEA configuration.†† Assumptions associated with analysis follow:

1) Cross coupling terms between INS axes are neglected (these errors typically are quite small). Similarly, cross axis gradiometer errors are not included.

2) Gradiometer errors are white noise only; low-frequency gradiometer errors are neglected. This assumption was made for convenience, as extension to other error sources, such as bias or time correlated noise, is simple.

3) The vehicle moves on or near the surface of the Earth.

4) The gravity gradient signal is "moving-window" averaged for 10 sec prior to output from the gradiometer.

5) Vertical deflections are described statistically by a second-order Markov process (Ref. 10). The deflection rms value is 8 arc sec and the deflection correlation distance is 20 nautical miles. These values correspond closely with those of the deflection models assumed in Ref. 1 and 9. The modeled gravity gradient rms value is 22.5 EU, with a correlation distance of 4.04 nautical miles.

6) The reference velocity error consists of white scale-factor noise, 0.2% of actual velocity with a 2-sec correlation time.

7) The value of the velocity damping constant ζ is 0.3. (The damping gain K_N given by $K_N = 2\zeta\omega_s$ is 4.5×10^{-3} min^{-1}.

8) Bias accelerometer error is 4 arc sec.

9) No other error sources (e.g., gyro drift rate) are present.

††Residual errors of both are the same for the single axis error equations. See discussion following Eq. (43).

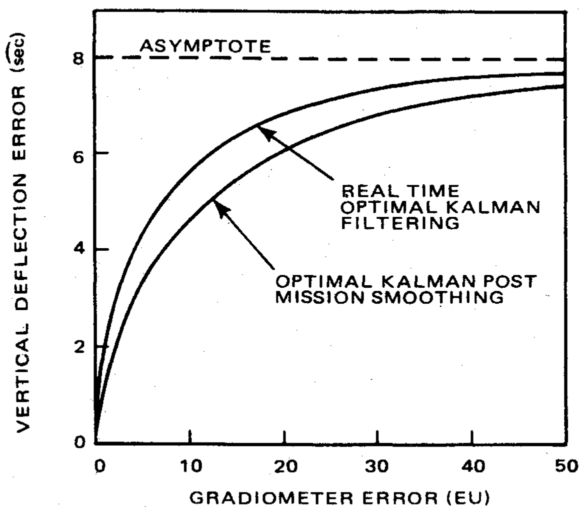

Fig.7a Steady-state rms vertical deflection vs gradiometer accuracy at 500 knot cruise speed.

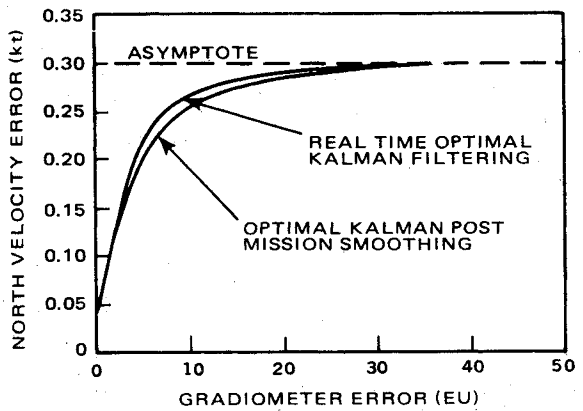

Fig.7b Steady-state rms velocity error vs gradiometer accuracy at 500 knot cruise speed.

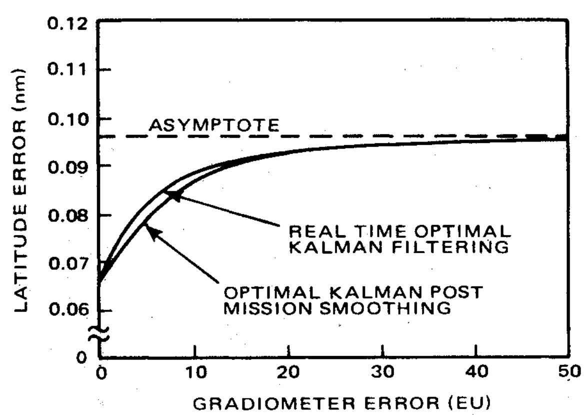

Fig.7c Steady-state rms latitude error vs gradiometer accuracy at 500 knot cruise speed.

Of course the effects of other independent sources easily could be included by taking root-sum-squares of the additional errors and those discussed herein.

10) No position fixes‡‡ or periodic gravity value inputs are used.

Equations used in the simulation were Eqs. (29) and the equations for a second-order Markov Process to represent

‡‡However, such known values of gravity could be combined with the gradiometer estimates. Gravity updates undoubtedly would be of value in reducing the filter initialization transient.

Table 1 Assumptions for gradiometer-aided INS simulations

rms Deflection = 8 arcsec (per axis)
Deflection correlation distance = 20 nautical miles
Gradiometer averaging time = 10 sec
Additional error sources are:
4 arcsec bias accelerometer error
0.2% V doppler scale factor fluctuation with 2 sec correlation time
Zero gyro drift rate

gravity. The single-axis versions of the damped INS error equstions [Eqs. (1) and (2)] are

$$\delta \dot{R}_N = \delta V_N \tag{44}$$

$$\delta \dot{V}_N = -\omega_s^2 \delta R_N - K_N(\delta V_N - \delta V_{rN}) + \delta A_N + \Delta g_N \tag{46}$$

where the subscript N indicates the north component of vector or matrix quantities already defined. These equations are depicted in Fig. 6. The Markov process frequency parameter β is defined by

$$\beta = 2.146 V_N / d \tag{46}$$

where d is the deflection correlation distance.

The results of two series of simulations for an aircraft flying at a horizontal groundspeed of 500 knots are presented in Figs. 7a-7c. Table 1 describes the key assumptions associated with Figs. 7a-7c. Both real-time Kalman filtering and post-mission Kalman smoothing are considered. Immediately apparent from Figs. 7a-7c are the stringent gradiometer error performance requirements at high velocity. In this example, 1 EU of gradiometer error corresponds to approximately 2 arcsec vertical deflection recovery accuracy with real-time optimal filtering; with post-mission smoothing the deflection estimates are correct to about 1 arcsec rms. With a gradiometer which is accurate to 1 EU, gravity-induced velocity and position errors nearly are eliminated.

The spectrum of vertical deflections varies significantly from one geographical region to the next. This variability suggests that: 1) an adaptive model may be needed in the Kalman filter and 2) the simulation results given here are not representative of system performance in all areas of the world. Thus, as improved models for gradiometer instruments errors are developed, similar improvements should be sought in vertical deflection models, and the potential benefits of adaptive filtering should be investigated.

VII. Summary and Conclusions

A general set of error equations is derived for two gradiometer-aided inertial navigation system configurations. On the basis of these equations it is concluded that operation of the gradiometer in the role of an external navigation aid (GAEA) is attractive, since all gravity-induced system errors are bounded quantities. Processing of the gradiometer and reference ellipsoid gravity information in a Kalman filter or a Kalman smoother provides optimal estimates of the gravity disturbances and the navigation errors.

Covariance simulation studies conducted on single-axis versions of these equations indicate that a vehicle, traveling at a constant velocity of 500 knots along a fixed course, obtains good navigation performance for gradiometer accuries near 1 EU (10 sec averaging time), when the gradiometer error is white noise.

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